

Groundstate with Zero Eigenvalue for Generalized Sombrero-shaped Potential in N -dimensional Space

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Abstract

Based on an iterative method for solving the groundstate of Schroedinger equation, it is found that a kind of generalized Sombrero-shaped potentials in N -dimensional space has groundstates with zero eigenvalue. The restrictions on the parameters in the potential are discussed.

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Recently G. 't Hooft et al.[1] discussed the possibility of the existence of a new kind of transformation from real to imaginary space-time variables. The invariance of this transformation needs to have zero eigenvalue for the groundstate. In this paper we discuss a kind of generalized Sombrero-shaped potentials in N -dimensional space and give restrictions to the parameters in the potential to give zero eigenvalue for the groundstate. The generalized Sombrero-shaped potentials in N -dimensional space are defined as

$$V(r) = \frac{1}{2}g^2(r^4 - \alpha r^2 + \beta)(r^2 + A) \quad (1)$$

where g^2 and α , β and A are taken to be arbitrary constants. The corresponding Schroedinger equation for the groundstate radial wave function is

$$\left(-\frac{1}{2r^{N-1}}\frac{d}{dr}r^{N-1}\frac{d}{dr} + V(r)\right)\psi(r) = E\psi(r) \quad (2)$$

The boundary conditions are

$$\psi(\infty) = 0 \quad \psi'(0) = 0. \quad (3)$$

In the following an iterative solution for the groundstate of (2) is given.

For a groundstate it is reasonable to express the wave function as

$$\psi(r) = e^{-S(r)}. \quad (4)$$

Substituting (4) into (2) an equation for $S(r)$ is obtained as

$$S'(r)^2 - \frac{N-1}{r}S'(r) - S''(r) = 2(V(r) - E) \quad (5)$$

An iterative method has been developed by Friedberg, Lee and Zhao[2] to solve this kind of problems. Following this iterative method, we introduce a trial function

$$\phi(r) = e^{-S_0(r)} \quad (6)$$

satisfying another Schroedinger equation

$$\left(-\frac{1}{2r^{N-1}}\frac{d}{dr}r^{N-1}\frac{d}{dr} + V(r) - h(r)\right)\phi(r) = E_0\phi(r) = (E - \Delta)\phi(r), \quad (7)$$

where $h(r)$ and Δ are the corrections of the potential and the eigenvalue of the groundstate. To ensure the convergency of the iterative method it is necessary to construct the trial function in such way that the perturbed potential $h(r)$ is positive (or negative) and finite everywhere. Specially, $h(r) \rightarrow 0$ when $r \rightarrow \infty$. Now substituting (6) into (7) we obtain the equation for $S_0(r)$:

$$S'_0(r)^2 - \frac{N-1}{r}S'_0(r) - S''_0(r) = 2(V(r) - h(r) - E_0). \quad (8)$$

Therefore

$$h(r) + E_0 = V(r) - \frac{1}{2}(S'_0(r)^2 - \frac{N-1}{r}S'_0(r) - S''_0(r)). \quad (9)$$

For a finite $h(r)$ it should not include terms with positive power of r . Since the highest order of r -power in the potential is 6 and the potential $V(r)$ has only even powers of r , for arbitrary normalization we can assume

$$S_0(r) = (ar^4 - cr^2) + m \log(r^2 + 1). \quad (10)$$

Substituting (10) into (8) to cancel r^6 term we have $a = g/4$ and

$$S_0(r) = (\frac{g}{4}r^4 - cr^2) + m \log(r^2 + 1). \quad (11)$$

To cancel the terms with r^4 and r^2 in (8) we get

$$4c = (\alpha - A)g, \quad (12)$$

$$4m = g(\beta - \alpha A) - \frac{1}{4}g(\alpha - A)^2 + N + 2. \quad (13)$$

and finally obtain

$$h(r) = 2m(m+1)\frac{1}{(r^2+1)^2} + ((N-2)m - 2m^2 - 2mg - 4mc)\frac{1}{r^2+1} \quad (14)$$

$$E_0 = \frac{1}{2}Ag^2\beta + 2mg - Nc + 4mc. \quad (15)$$

When we set $m = 0$, i.e. $h = 0$, the trial function becomes the exact groundstate wave function:

$$\phi(r) = e^{-\frac{g}{4}r^4 + cr^2} \quad (16)$$

with eigenvalue

$$E_0 = \frac{1}{2}Ag^2\beta - Nc. \quad (17)$$

This gives the following relation between the parameters:

$$g(\beta - \alpha A) - \frac{1}{4}g(\alpha - A)^2 + N + 2 = 0. \quad (18)$$

Further introducing $E_0 = 0$, i.e.

$$\frac{1}{2}Ag^2\beta - Nc = 0 \quad (19)$$

(12), (18) and (19) give the condition to the parameters for a groundstate with zero eigenvalue.

When taking $c = 0$, i.e. $A = \alpha$, we have

$$E_0 = \frac{1}{2}g^2A\beta \quad (20)$$

and there are two possible ways to give zero eigenvalue. For $A = 0$ we have

$$V(r) = \frac{1}{2}g^2(r^4 + \beta)r^2 \quad (21)$$

with its only minimum at $r = 0$. For $\beta = 0$, we have

$$V(r) = \frac{1}{2}g^2r^2(r^4 - A^2) \quad (22)$$

with its minima at $r = 0$ and $r^2 = A$. However, in both cases the ground-state wave function is

$$\phi(r) = e^{-\frac{1}{4}gr^4} \quad (23)$$

with its only maximum at $r = 0$. Only when we take $c > 0$, i.e. $A < \alpha$, the wave function can have more than one maxima at $r^2 = 2c/g$. For $c > 0$ to ensure $E_0 = 0$ we must have

$$g^2A\beta = 2Nc > 0. \quad (24)$$

Introducing $A = \eta\alpha$ and $\alpha^2 = \lambda 4\beta$ the condition of $m = 0$ and $E_0 = 0$ gives the following two equations:

$$g\beta(1 - \lambda(1 + \eta)^2) + N + 2 = 0 \quad (25)$$

$$g\beta = \frac{1}{2}N\frac{1-\eta}{\eta} \quad (26)$$

which give a relation between η and λ :

$$\lambda N\eta^3 + \lambda N\eta^2 + (N + 4 - \lambda N)\eta + N(1 - \lambda) = 0. \quad (27)$$

As an example taking $N = 3$ the function $\eta(\lambda)$ is plotted in Fig. 1 for $\lambda > 1$ and $0 < \eta < 1$. For any chosen g and λ the other parameters η , β , α and A can all be fixed to ensure $m = 0$ and $E_0 = 0$. For the 3-dimensional case, if choosing $g = 1.5$ and $\lambda = 1.5$, from (26) and (27) the parameters are obtained as $\eta = 1/3$, $\beta = 2$, $\alpha = \sqrt{12}$ and $A = \sqrt{12}/3$. This gives $c = \sqrt{3}/2$. The groundstate wave function is

$$\phi(r) = e^{-\frac{3}{8}r^4 + \frac{\sqrt{3}}{2}r^2} \quad (28)$$

which has maxima at $r^2 = 2c/g = 2/\sqrt{3}$ and a valley at $r = 0$. In Figs. 2 and 3 the potential and the wave function are plotted respectively. It is interesting to obtain these analytic solutions of the groundstate wave function, which are degenerate and with zero eigenvalue, by choosing special sets of parameters for the potential.

As an example discussed by R. Jackiw one can introduce a parameter

$$r_0^4 = \frac{N + 2}{3}. \quad (29)$$

When $g = 1$, $\beta = r_0^4$ and $A = 2r_0^2$ one can choose special values of α , which give $m = 0$, to obtain analytic solutions of the groundstate directly from the trial functions. In this set of parameters the condition of $m = 0$ gives the following equation:

$$\alpha^2 + 4r_0^2\alpha - 12r_0^4 = 0 \quad (30)$$

which gives two solutions for α : When taking $\alpha = 2r_0^2$ we have $c = 0$ and the groundstate is

$$\psi(r) = e^{-r^4/4}, \quad E_0 = r_0^6, \quad (31)$$

which is the solution given by R. Jackiw in [3]; while $\alpha = -6r_0^2$ gives $c = -2r_0^2$ and the solution is

$$\psi(r) = e^{-r^4/4 - 2r_0^2r^2}, \quad E_0 = r_0^6 + 2Nr_0^2. \quad (32)$$

If we write the potential in the form of

$$V(r) = \frac{1}{2}g^2[(r^2 - r_0^2)^2 - \mu r_0^4](r^2 - 2\eta r_0^2), \quad (33)$$

comparing (33) with (1) the parameters are related in the following way:

$$\alpha = 2r_0^2, \quad \beta = r_0^4(1 - \mu), \quad A = \eta\alpha. \quad (34)$$

To obtain an analytical solution for the groundstate with zero eigenvalue, from (12), (18) and (19) we have

$$c = \frac{1}{2}(1 - \eta)gr_0^2, \quad (35)$$

$$g(1 - \mu) - g(1 + \eta)^2 + 3 = 0 \quad (36)$$

and

$$\eta g^2(1 - \mu)r_0^6 - Nc = 0. \quad (37)$$

The wave function is expressed as

$$\phi(r) = e^{-\frac{g}{4}r^4 + cr^2} \quad (38)$$

For $N = 3$, taking $g = 1$, substituting (35) and (36) into (37) an equation for η is obtained as following:

$$2\eta^3 + 4\eta^2 - \frac{11}{5}\eta - \frac{9}{5} = 0. \quad (39)$$

From (39) and (36) we find values of η and μ , which give $m = 0$ and $E_0 = 0$, i.e. the analytical groundstate with zero eigenvalue:

$$\eta = 0.797005, \quad \mu = 0.770765,$$

correspondingly, from (35)

$$c = 0.103152.$$

The wave function has degenerate groundstate with zero eigenvalue and its maxima at $r = \sqrt{2c/g} = 0.4542$.

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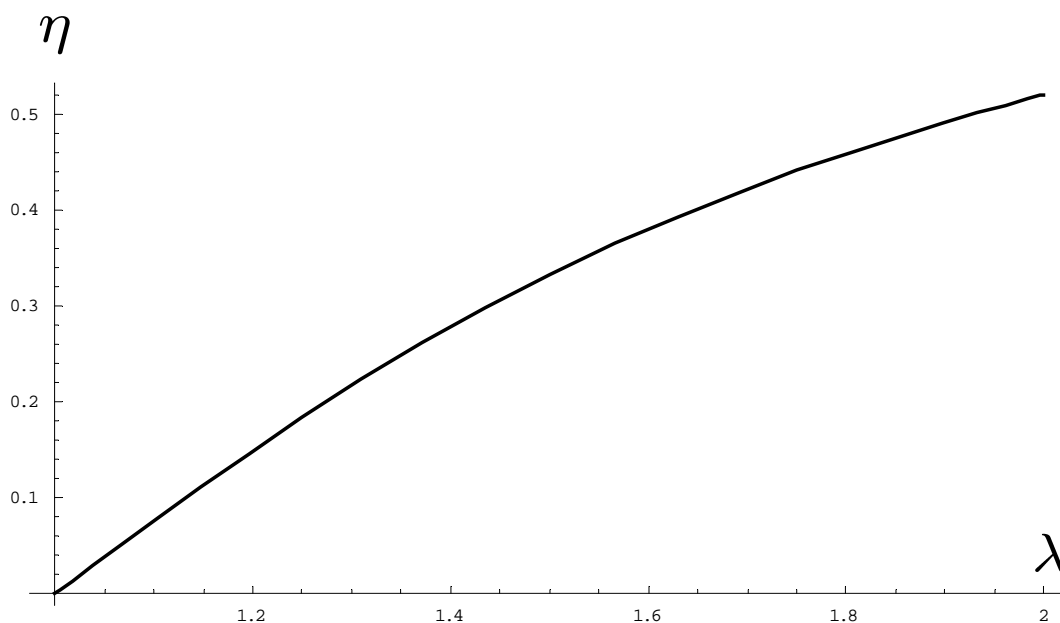


Fig. 1 $\eta(\lambda)$ for the groundstate with zero eigenvalue

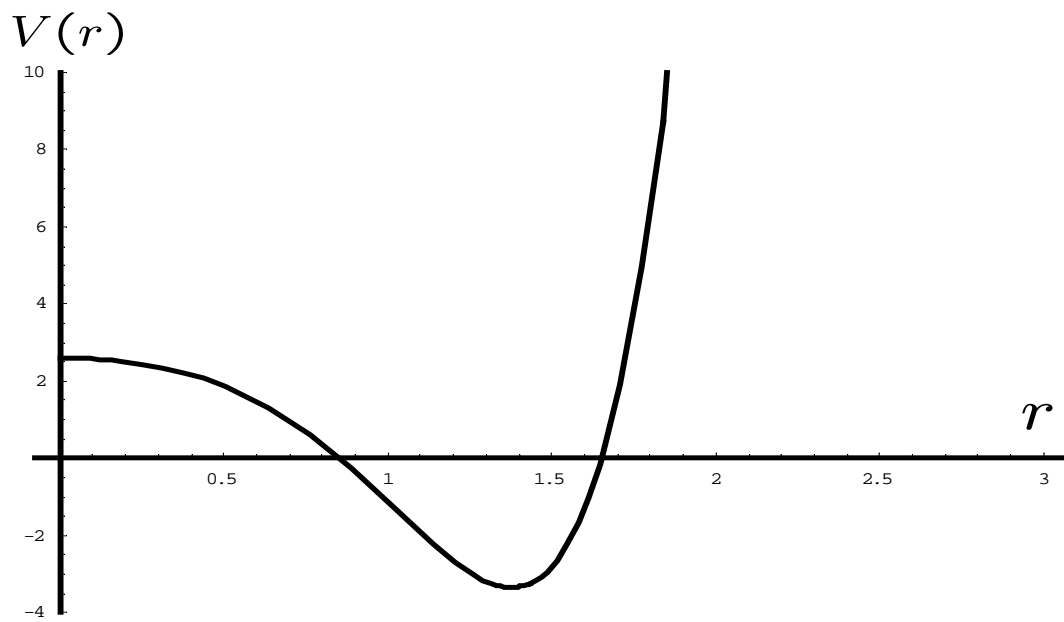


Fig. 2 $V(r)$ for $N = 3$, $g = 1.5$ and $\lambda = 1.5$

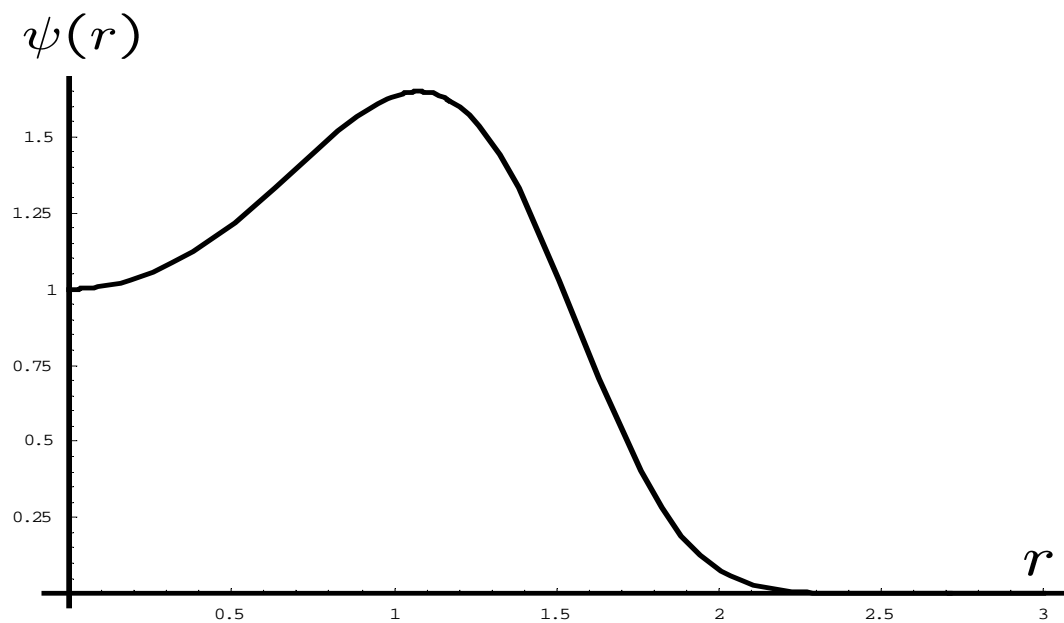


Fig. 3 $\psi(r)$ for $N = 3$, $g = 1.5$ and $\lambda = 1.5$